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UNICOHERENCE AT SUBCONTINUA

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The notion of unicoherence at a subcontinuum of a metric continuum is defined, and relationships between this local form of unicoherence and certain existing forms of unicoherence are investigated. In particular, it is shown that strong unicoherence is equivalent to unicoherence at every subcontinuum and that weak hereditary unicoherence is equivalent to unicoherence at every subcontinuum having nonempty interior. Further, if every indecomposable subcontinuum of a continuum X has nonempty interior, then X is strongly unicoherent if and only if X is weakly hereditarily unicoherent. New characterizations of dendrites are obtained by requiring either aposyndesis or local connectivity and unicoherence at certain subcontinua.

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hereditarily unicoherent
strongly unicoherent
unicoherent at a subcontinuum
weakly hereditarily unicoherent
aposyndetic
dendrite
unicoherent

Introduction

Let X denote a metric continuum. In this paper, the following localization of the notion of unicoherence is introduced: A continuum X is *unicoherent at a subcontinuum* Y if for each pair of proper subcontinua A and B of X having union X the set $A \cap B \cap Y$ is connected. Note that if $A \cap B \cap Y$ is empty then the set is connected. Several fundamental results concerning unicoherence at a subcontinuum are derived in Section 1.

The main purpose of this paper is to use the results in Section 1 to obtain characterizations of certain kinds of unicoherence which appear in the literature. Several characterizations of strong unicoherence, defined by Bennett [1], are obtained; the most important of these characterizations is that strong unicoherence is equivalent to unicoherence at every subcontinuum. Maćkowiak's notion of weak hereditary unicoherence [5] is characterized by requiring unicoherence at every subcontinuum having nonempty interior. The characterizations of strong unicoherence and weak hereditary unicoherence are then used to prove that if every indecomposable subcontinuum of a continuum has nonempty interior it follows that strong unicoherence is equivalent to weak hereditary unicoherence; this result generalizes a theorem of Maćkowiak's [5].

Bennett has proved that a metric continuum is a dendrite if and only if it is strongly unicoherent and aposyndetic [1]. In Section 3, new characterizations of dendrites are obtained by requiring unicoherence at certain subcontinua; in particular, the results in Section 2 are used to prove that a continuum X is a dendrite if and only if X is weakly hereditarily unicoherent and aposyndetic, generalizing Bennett's characterization.

1. Preliminary results

Throughout this paper, a *continuum* is a compact connected metric space. A continuum X is *unicoherent* if the intersection of every two subcontinua having union X is connected; a continuum X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent. Let Y be a subcontinuum of X; X is *unicoherent* at Y, denoted $\mathrm{Un}(Y)$, if for each pair of proper subcontinua A and B of X such that $X = A \cup B$ the set $A \cap B \cap Y$ is connected. The first two propositions follow immediately from the definitions.

Proposition 1.1. *A continuum X is unicoherent if and only if X is* $\mathrm{Un}(X)$.

Proposition 1.2. *If the continuum X is hereditarily unicoherent, then X is unicoherent at each of its subcontinua.*

The converse of Proposition 1.2 does not hold; if X is the continuum consisting of a ray limiting on a circle, then X is unicoherent at each of its subcontinua, but X is not hereditarily unicoherent because X contains a circle.

Proposition 1.3. *Let Y be a subcontinuum of a continuum X. If X is unicoherent at every irreducible subcontinuum of Y, then X is* $\mathrm{Un}(Y)$.

Proof. If X is not $\mathrm{Un}(Y)$, then there exist proper subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap Y$ is not connected. Since $A \cap B \cap Y$ is not connected, there exist nonempty disjoint closed subsets P and Q such that $A \cap B \cap Y = P \cup Q$. If $p \in P$ and $q \in Q$, then the points p and q lie in Y; therefore, there exists a subcontinuum I of Y that is irreducible between p and q. Since $A \cap B \cap I = (P \cap I) \cup (Q \cap I)$, a union of nonempty disjoint closed sets, X is not $\mathrm{Un}(I)$. □

The converse of Proposition 1.3 fails to hold. Let X be the continuum lying in the plane consisting of a ray limiting on a circle with an arc attached to the circle so that the ray and the arc are disjoint; in polar coordinates,

$$X = \{(2, \theta): 0 \leq \theta < 2\pi\} \cup \{(r, \theta): r = 2 - 1/\theta,\ \theta \geq 1\} \cup \{(r, 0): 2 \leq r \leq 3\}.$$

If

$$Y=\{(2,\theta): 0\leq\theta<2\pi\}\cup\{(r,0): 2\leq r\leq 3\},$$

then X is $\mathrm{Un}(Y)$; however, X is not unicoherent at

$$I=\{(2,\theta): 0\leq\theta\leq\pi\}\cup\{(r,0): 2\leq r\leq 3\},$$

an irreducible subcontinuum of Y.

The property of unicoherence at a subcontinuum is additive in the sense of the following theorem.

Theorem 1.4. *Let Y and Z be subcontinua of a continuum X such that X is* $\mathrm{Un}(Y)$, *X is* $\mathrm{Un}(Z)$, *and $Y\cap Z\neq\emptyset$. Then X is* $\mathrm{Un}(Y\cup Z)$.

Proof. Suppose X is $\mathrm{Un}(Y)$, $\mathrm{Un}(Z)$, and not $\mathrm{Un}(Y\cup Z)$. There exist proper subcontinua A and B of X such that $X=A\cup B$ and the set

$$A\cap B\cap(Y\cup Z)=(A\cap B\cap Y)\cup(A\cap B\cap Z)$$

is not connected. Since X is $\mathrm{Un}(Y)$ and $\mathrm{Un}(Z)$, each of the sets $A\cap B\cap Y$ and $A\cap B\cap Z$ is connected. It follows that both $A\cap B\cap Y$ and $A\cap B\cap Z$ are nonempty; moreover,

$$(A\cap B\cap Y)\cap(A\cap B\cap Z)=A\cap B\cap Y\cap Z=\emptyset.$$

Without loss of generality, there exists a point x in $A\cap Y\cap Z$. Consider $X=A\cup(B\cup Z)$. Since $A\cap B\cap Z$ is nonempty, the set $B\cap Z$ is nonempty and $B\cup Z$ is a subcontinuum of X. If $B\cup Z$ is a proper subcontinuum of X, then the unicoherence of X at Y implies that the set

$$A\cap(B\cup Z)\cap Y=(A\cap B\cap Y)\cup(A\cap Y\cap Z)$$

is connected. Note that $A\cap B\cap Y$ and $A\cap Y\cap Z$ are closed and nonempty; however,

$$(A\cap B\cap Y)\cap(A\cap Y\cap Z)=A\cap B\cap Y\cap Z=\emptyset$$

which contradicts the connectivity of $A\cap(B\cup Z)\cap Y$. It follows that $B\cup Z=X$. Therefore, $Y=(B\cap Y)\cup(Z\cap Y)$, and since $X=A\cup B$,

$$\begin{aligned}Y&=(B\cap Y)\cup(A\cap Z\cap Y)\cup(B\cap Z\cap Y)\\&=(B\cap Y)\cup(A\cap Z\cap Y).\end{aligned}$$

Since $A\cap B\cap Y$ is nonempty, the set $B\cap Y$ is closed and nonempty; once again, $x\in A\cap Z\cap Y$ implies that $A\cap Z\cap Y$ is closed and nonempty. However,

$$(B\cap Y)\cap(A\cap Z\cap Y)=A\cap B\cap Y\cap Z=\emptyset,$$

and this contradicts the connectivity of Y.

Therefore, X is $\mathrm{Un}(Y\cup Z)$. □

Theorem 1.4 easily generalizes to the case of a union of finitely many subcontinua as stated in the following corollary; details of the proof are omitted.

Corollary 1.5. *Let $Y_1, Y_2, \dots, Y_n$ be a finite collection of subcontinua of a continuum X such that X is* $\mathrm{Un}(Y_i)$ *for every i, and suppose that for each $i > 1$*

$$Y_i \cap \bigcup\{Y_j : j < i\} \neq \emptyset.$$

Then X is $\mathrm{Un}(\bigcup_{i=1}^{n} Y_i)$.

The next two technical results will be needed in the following sections.

Theorem 1.6. *Let Y be a subcontinuum of a continuum X. If X is* $\mathrm{Un}(Y)$ *and A and B are proper subcontinua of X such that $X = A \cup B$, then the sets $A \cap Y$ and $B \cap Y$ are connected.*

Proof. If the set $A \cap Y$ is not connected, then there exist closed nonempty disjoint sets P and Q so that $A \cap Y = P \cup Q$. Since X is $\mathrm{Un}(Y)$, the set $A \cap B \cap Y$ is a connected subset of $P \cup Q$; therefore, without loss of generality, it may be assumed that $A \cap B \cap Y$ lies in P. If the set $Q \cap B$ is nonempty, then there exists a point q in Q, a subset of $A \cap Y$, such that $q \in A \cap B \cap Y \subseteq P$, a contradiction; therefore, the set $Q \cap B$ is empty.

Let I be a subcontinuum of Y that is irreducible between the sets P and Q; note that

$$I \setminus (P \cup Q) \subseteq Y \setminus (P \cup Q) = Y \setminus A \subseteq Y \cap B \subseteq B,$$

and it follows that

$$(I \cap P) \cup (I \cap Q) \subseteq \overline{I \setminus (P \cup Q)} \subseteq B.$$

Therefore, the subcontinuum I is contained in B; consequently, the set $B \cap Q$ is nonempty, a contradiction. Therefore, the set $A \cap Y$ is connected, and similarly, $B \cap Y$ is connected. □

For a subset S of X, let S^0 denote the interior of S in X.

Theorem 1.7. *Let H and K be subcontinua of a continuum X such that $H^0 \neq \emptyset$, X is* $\mathrm{Un}(H)$, *and $H \cap K$ is not connected. Then X is not* $\mathrm{Un}(K)$.

Proof. Consider the set $X \setminus H$.

Case 1. *The set $X \setminus H$ is not connected.* There exist nonempty separated sets M and N such that $X \setminus H = M \cup N$. Note that $\bar{M} \cap \bar{N} \subseteq H$, the sets $H \cup \bar{M}$ and $H \cup \bar{N}$ are proper subcontinua of X, and $X = (H \cup \bar{M}) \cup (H \cup \bar{N})$. Since

$$(H \cup \bar{M}) \cap (H \cup \bar{N}) \cap K = [H \cup (\bar{M} \cap \bar{N})] \cap K = H \cap K$$

and $H \cap K$ is not connected, the continuum X is not $\mathrm{Un}(K)$.

Case 2. *The set $X \setminus H$ is connected.* It suffices to show that the set $(\overline{X \setminus H}) \cup K$ is a proper subcontinuum of X; if this is the case, then $X = H \cup [(\overline{X \setminus H}) \cup K]$ and $H \cap [(\overline{X \setminus H}) \cup K] \cap K = H \cap K$ is not connected. Therefore, X is not $\mathrm{Un}(K)$.

If $(X\backslash H)\cap K=\emptyset$, then $K\subseteq H$ and $H\cap K=K$ is connected, a contradiction; therefore, $(X\backslash H)\cap K\neq\emptyset$, and $(\overline{X\backslash H})\cup K$ is a subcontinuum of X. Now suppose that $(\overline{X\backslash H})\cup K$ is not a propert subcontinuum of X; then $\overline{H^0}\subseteq K$. Since $H\cap K$ is not connected, K is a proper subcontinuum of X, and since $H^0\neq\emptyset$, the set $\overline{X\backslash H}$ is a proper subcontinuum of X. It follows from the unicoherence of X at H that

$$(\overline{X\backslash H})\cap K\cap H=\partial H\cap K$$

is connected. Consider the set

$$H\cap K=(\overline{H^0}\cup\partial H)\cap K=(\overline{H^0}\cap K)\cup(\partial H\cap K)=\overline{H^0}\cup(\partial H\cap K).$$

Let C be a component of $\overline{H^0}$; then C is closed. If $C\cap\partial(H^0)=\emptyset$, then $C\subseteq H^0$ and C is a component of H^0; therefore, $\bar{C}\cap\partial(H^0)=C\cap\partial(H^0)\neq\emptyset$, a contradiction. Consequently, $C\cap\partial(H^0)$ is not empty. Since $\partial(H^0)\subseteq\partial H$, it follows that $C\cap\partial H\neq\emptyset$, and since $C\subseteq\overline{H^0}\subseteq K$, the set $C\cap\partial H\cap K$ is not empty. Therefore, each component of $\overline{H^0}$ meets the connected set $\partial H\cap K$, and $\overline{H^0}\cup(\partial H\cap K)$ must be connected. From this contradiction, it follows that $(\overline{X\backslash H})\cup K$ is a proper subcontinuum of X. □

2. Characterizations of certain kinds of unicoherence

The notion of strong unicoherence was introduced by Bennett in [1]. A unicoherent continuum X is *strongly unicoherent* if for every pair of proper subcontinua A and B such that $X=A\cup B$ both A and B are unicoherent. Recall the two examples of Section 1; the continuum consisting of a ray limiting on a circle is strongly unicoherent, and the continuum consisting of a ray limiting on a circle with an arc attached is not strongly unicoherent. Several characterizations of strong unicoherence in terms of unicoherence at subcontinua are possible.

Theorem 2.1. *The continuum X is strongly unicoherent if and only if X is unicoherent at each of its subcontinua.*

Proof. Suppose X is strongly unicoherent and Y is a subcontinuum of X such that X is not $\mathrm{Un}(Y)$. There exist proper subcontinua A and B of X such that $X=A\cup B$ and $A\cap B\cap Y$ is not connected. Since $A\cap B\cap Y$ is not connected, $B\cap Y$ is nonempty. Note that $X=A\cup(B\cup Y)$, and either $B\cup Y=X$ or $B\cup Y$ is a proper subcontinuum of X; in each case, the strong unicoherence of X implies that $B\cup Y$ is unicoherent. Therefore, $B\cap Y$ is connected, amd since $A\cap B\cap Y$ is not connected, $A\cup(B\cap Y)$ is a nonunicoherent subcontinuum of X. Consider $X=[A\cup(B\cap Y)]\cup B$. Either $X=A\cup(B\cap Y)$ or $A\cup(B\cap Y)$ is a proper subcontinuum of X; since X is strongly unicoherent, a contradiction arises in each case. Therefore, strong unicoherence implies unicoherence at every subcontinuum.

Conversely, suppose X is unicoherent at every subcontinuum and X is not strongly unicoherent. Since X is $\mathrm{Un}(X)$, X is unicoherent by Proposition 1.1; therefore, there exist proper subcontinua A and B of X such that $X = A \cup B$ and A is not unicoherent. Let H and K be proper subcontinua of A such that $A = H \cup K$ and $H \cap K$ is not connected. Since $A \cap B \neq \emptyset$, without loss of generality, it may be assumed that $B \cap H$ is nonempty. Therefore, $B \cup H$ is a subcontinuum of X; moreover, the set $(B \cup H) \cup K$ is equal to X. If $B \cup H$ is a proper subcontinuum of X, then the unicoherence of X at H implies that

$$(B \cup H) \cap K \cap H = (B \cap K \cap H) \cup (H \cap K) = H \cap K$$

is connected, a contradiction. Therefore, suppose $B \cup H = X$ and note that $B \cap K$ is nonempty. It follows that $B \cup K$ is a subcontinuum of X and $X = (B \cup K) \cup H$. Now if $B \cup K$ is a proper subcontinuum of X, then the unicoherence of X at K gives rise to a contradiction.

Consequently,

$$X = B \cup H = B \cup K$$

and

$$X \setminus B = H \setminus B = K \setminus B;$$

therefore, $X \setminus B$ is a nonempty subset of $H \cap K$. Since $X = B \cup H$ and X is $\mathrm{Un}(K)$, $B \cap H \cap K$ is connected; since $H \cap K$ is not connected, there is some component C of $H \cap K$ such that $C \cap (B \cap H \cap K) = \emptyset$. It follows that

$$C \subseteq X \setminus B \subseteq H \cap K,$$

and C is a component of $X \setminus B$; therefore, $\bar{C}$ meets the boundary of B and $\bar{C} \cap B \neq \emptyset$. Since $H \cap K$ is closed, $\bar{C} = C$ and hence $C \cap B \neq \emptyset$. This contradiction establishes the theorem. □

The next corollary follows directly from the theorem and Proposition 1.3.

Corollary 2.2. *The continuum X is strongly unicoherent if and only if X is unicoherent at each of its irreducible subcontinua.*

The preceding corollary indicates that unicoherence at every member of a certain class of subcontinua can force unicoherence at every subcontinuum; the following theorem is another result in this vein.

Theorem 2.3. *A nececessary and sufficient condition for the strong unicoherence of a continuum X is that X is* $\mathrm{Un}(Y)$ *for every subcontinuum Y such that $Y^0 \neq \emptyset$ or Y is indecomposable.*

Proof. The necessity of the condition is an immediate consequence of Theorem 2.1.

On the contrary, suppose the condition is not sufficient. If X is not strongly unicoherent, then by Corollary 2.2, there exists an irreducible subcontinuum I of X such that X is not Un(I). Since X is not Un(I), there exist proper subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap I = P \cup Q$, a union of nonempty disjoint closed sets. Let J be a subcontinuum of I that is irreducible between P and Q. Since $A \cap B \cap J$ is contained in $P \cup Q$ and meets both P and Q, $A \cap B \cap J$ is not connected; therefore, X is not Un(J). It follows that $J^0 = \emptyset$ and J is decomposable. Suppose J_P and J_Q are proper subcontinua of J such that $J = J_P \cup J_Q$, $J_P \cap Q = \emptyset$, and $J_Q \cap P = \emptyset$; note that $A \cap B \cap J_P \cap J_Q = \emptyset$. Let $x \in J_P \cap J_Q$; without loss of generality, it may be assumed that x lies in $A \backslash B$.

Consider $B \cup J_P$, a subcontinuum of X. Since $J^0 = \emptyset$, it follows that $J_P^0 = \emptyset$; therefore, if $X = B \cup J_P$, then $J_P \subseteq \overline{X \backslash J_P} \subseteq B$ which is a contradiction since $x \in J_P \backslash B$. Consequently, $B \cup J_P$ must be a proper subcontinuum of X. Note that $B \cup J_Q$ has nonempty interior since $B^0 \neq \emptyset$; therefore, X is Un($B \cup J_Q$). Since $X = A \cup (B \cup J_P)$, it follows that the set

$$\begin{aligned} A \cap (B \cup J_P) \cap (B \cup J_Q) &= A \cap [B \cup (J_P \cap J_Q)] \\ &= (A \cap B) \cup (A \cap J_P \cap J_Q) \end{aligned}$$

is connected; however, $A \cap B$ and $A \cap J_P \cap J_Q$ are nonempty disjoint closed sets, a contradiction. □

The final characterization of strong unicoherence in this section is due to Maćkowiak [5]; the proof given below involves the characterization of strong unicoherence in Theorem 2.1.

Theorem 2.4. *The continuum X is strongly unicoherent if and only if every subcontinuum of X having nonempty interior is unicoherent.*

Proof. Suppose the continuum X is strongly unicoherent and Y is a nonunicoherent subcontinuum of X having nonempty interior. Then there exist proper subcontinua H and K of Y such that $Y = H \cup K$, $H^0 \neq \emptyset$, and $H \cap K$ is not connected. By Theorem 2.1, X is unicoherent at each of its subcontinua, and in particular, X is Un(H) and Un(K). However, by Theorem 1.7, X is not Un(K), a contradiction.

For the converse, suppose X is not strongly unicoherent and every subcontinuum of X having nonempty interior is unicoherent. By Theorem 2.1, there exists a subcontinuum Y of X such that X is not Un(Y). Let A and B be proper subcontinua of X such that $X = A \cup B$ and $A \cap B \cap Y$ is not connected; then both A and B have nonempty interiors and $A \cap B \cap Y$ is nonempty. It follows that $B \cap Y \neq \emptyset$, and $B \cup Y$ is a subcontinuum of X having nonempty interior. Therefore, $B \cup Y$ is unicoherent, and $B \cap Y$ is connected. Since both A^0 and $A \cap B \cap Y$ are nonempty, it follows that the set $A \cup (B \cap Y)$ is a nonunicoherent subcontinuum of X having nonempty interior, a contradiction. □

A continuum X is *weakly hereditarily unicoherent* if the intersection of every two subcontinua of X having nonempty interiors is connected; this type of unicoherence was defined by Maćkowiak in [4]. The characterization of weak hereditary unicoherence which appears in the next theorem will be used to investigate the relationship between strong unicoherence and weak hereditary unicoherence.

Theorem 2.5. *The continuum X is weakly hereditarily unicoherent if and only if X is unicoherent at every subcontinuum having nonempty interior.*

Proof. Suppose X is weakly hereditarily unicoherent, and let Y be a subcontinuum of X having nonempty interior. If A and B are proper subcontinua of X such that $X = A \cup B$, then both A and B have nonempty interiors. It follows that both $A \cap Y$ and $B \cap Y$ are connected. Since $Y = (A \cap Y) \cup (B \cap Y)$ and $Y^0 \neq \emptyset$, it may be assumed that $(A \cap Y)^0 \neq \emptyset$. Consequently, $A \cap Y$ is a subcontinuum of X having nonempty interior; the weak hereditary unicoherence of X implies that $(A \cap Y) \cap B$ is connected. Therefore, X is $\mathrm{Un}(Y)$.

Conversely, assume that X is unicoherent at every subcontinuum having nonempty interior. If X is not weakly hereditarily unicoherent, then there exist subcontinua H and K of X such that $H^0 \neq \emptyset$, $K^0 \neq \emptyset$, and $H \cap K$ is not connected. Since both H and K have nonempty interiors, X is $\mathrm{Un}(H)$ and X is $\mathrm{Un}(K)$. However, H and K satisfy the hypotheses of Theorem 1.7; therefore, X is not $\mathrm{Un}(K)$, a contradiction. □

The following corollary is an immediate consequence of Theorem 2.1 and Theorem 2.5.

Corollary 2.6. *If the continuum X is strongly unicoherent, then X is weakly hereditarily unicoherent.*

The implication in Corollary 2.6 was proved by Maćkowiak in [5] by using the characterization of strong unicoherence in Theorem 2.4; in the same paper, Maćkowiak constructs an example involving indecomposable subcontinua which demonstrates that the converse of Corollary 2.6 does not hold. Further, Maćkowiak proves that strong unicoherence is equivalent to weak hereditary unicoherence in a hereditarily decomposable continuum; Corollary 2.7 is a generalization of this proposition and follows directly from Theorem 2.3 and Theorem 2.5.

Corollary 2.7. *Suppose every indecomposable subcontinuum of a continuum X has nonempty interior in X. Then X is strongly unicoherent if and only if X is weakly hereditarily unicoherent.*

3. Characterizations of dendrites

A *dendrite* is a locally connected continuum that contains no simple closed curves. One of the many characterizations of dendrites is due to Bennett [1] and involves the notions of strong unicoherence and aposyndesis.

A continuum X *is aposyndetic at the point* p if for every q in $X\backslash\{p\}$ there exists a subcontinuum Y such that $p \in Y^0 \subseteq Y \subseteq X\backslash\{q\}$. A continuum X is *aposyndetic* if X is aposyndetic at each of its points. For continua, local connectivity clearly implies aposyndesis; however, it is well known that the two concepts are not equivalent.

Before proceeding with the characterizations of dendrites, the following observation dealing with the consequences of aposyndesis and unicoherence at a subcontinuum is made.

Theorem 3.1. *Let Y be a nondegenerate subcontinuum of a continuum X. If X is aposyndetic and* Un(Y), *then Y is decomposable.*

Proof. Let a and b be distinct points in Y. Since X is aposyndetic, there exist proper subcontinua A and B of X such that $a \in A\backslash B$, $b \in B\backslash A$, and $X = A \cup B$ [2]. By Theorem 1.6, the unicoherence of X at Y implies that $A \cap Y$ and $B \cap Y$ are connected; moreover, $Y = (A \cap Y) \cup (B \cap Y)$. Since a does not lie in $B \cap Y$ and b does not lie in $A \cap Y$, the sets $A \cap Y$ and $B \cap Y$ are proper subcontinua of Y; therefore, Y is decomposable. □

In [1], Bennett proves that a continuum X is a dendrite if and only if X is aposyndetic and strongly unicoherent. In the following theorem and corollary, Bennett's hypotheses of aposyndesis and strong unicoherence are replaced by requiring local connectivity and unicoherence at certain subcontinua.

Theorem 3.2. *The continuum X is a dendrite if and only if for each $\varepsilon > 0$ X can be covered by finitely many subcontinua $Y_1, Y_2, \ldots, Y_n$ such that each Y_i has diameter less than ε and X is unicoherent at each Y_i.*

Proof. If X is a dendrite, then X is hereditarily unicoherent and it follows that X is unicoherent at each of its subcontinua; the existence of the desired cover for X follows from Theorem 2, page 256, in [3].

Conversely, by Bennett's theorem and Theorem 2.1, it suffices to show that X is unicoherent at each of its subcontinua; suppose this is not the case. Let Z be a subcontinuum of X such that X is not Un(Z); then there exist proper subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap Z = P \cup Q$, a union of nonempty disjoint closed sets. Let $3\varepsilon_1 = \operatorname{dist}(P, Q)$; then $\varepsilon_1 > 0$. Cover $P \cup Q$ with finitely many open sets $U_1, U_2, \ldots, U_m$ such that for each i $\operatorname{diam}(U_i) < \varepsilon_1$ and $U_i \cap (P \cup Q) \neq \emptyset$. Let

$$U_P = \bigcup \{U_i\colon\ U_i \cap P \neq \emptyset\}$$

and let

$$U_Q = \bigcup \{U_i\colon U_i \cap Q \neq \emptyset\}.$$

Since $\operatorname{diam}(U_i) < \varepsilon_1$ for all i, it follows that $U_P \cap U_Q = \emptyset$. Let $U = U_P \cup U_Q$.

Consider the subcontinuum Z. If $Z \subseteq U$, then $Z = (Z \cap U_P) \cup (Z \cap U_Q)$, a union of disjoint sets which are open in Z. Since Z is connected, it may be assumed that $Z \cap U_P = \emptyset$; however, it then follows that $Z = Z \cap U_Q$. Consequently, $P \subseteq U_Q$, but this contradicts the fact that $P \cap U_Q = \emptyset$. Therefore, the set $Z \setminus U$ is nonempty.

Note that $Z \setminus U = Z \cap (X \setminus U)$ is closed in X. Let

$$\varepsilon = \operatorname{dist}(Z \setminus U, A \cap B);$$

then $\varepsilon > 0$ and $\varepsilon < \varepsilon_1$. Since $\varepsilon > 0$, there exists a cover of X as in the statement of the theorem; therefore, there is a subcover $Y = \bigcup_{i=1}^{k} Y_i$ of Z such that for each $i = 1, 2, \ldots, k$, $Y_i \cap Z \neq \emptyset$, $\operatorname{diam}(Y_i) < \varepsilon$, and X is $\operatorname{Un}(Y_i)$. By Corollary 1.5, X is $\operatorname{Un}(Y)$.

Since Y covers Z and $A \cap B \cap Z \neq \emptyset$, it follows that $A \cap B \cap Y \neq \emptyset$. If $x \in A \cap B \cap Y$, then there exists Y_h such that $1 \leq h \leq k$ and $x \in Y_h$. Since $Y_h \cap A \cap B \neq \emptyset$ and $\operatorname{diam}(Y_h) < \varepsilon$, the set $Y_h \cap (Z \setminus U)$ is empty; moreover, $Y_h \cap Z \neq \emptyset$ implies that $Y_h \cap Z \cap U \neq \emptyset$. Therefore $Y_h \cap U \neq \emptyset$. It follows that

$$\begin{aligned} &A \cap B \cap Y \\ &\quad = A \cap B \cap (\bigcup \{Y_i\colon Y_i \cap U_P \neq \emptyset\} \cup \bigcup \{Y_i\colon Y_i \cap U_Q \neq \emptyset\}) \\ &\quad = [A \cap B \cap (\bigcup \{Y_i\colon Y_i \cap U_P \neq \emptyset\})] \cup [A \cap B \cap (\bigcup \{Y_i\colon Y_i \cap U_Q \neq \emptyset\})]. \end{aligned}$$

Since $\operatorname{diam}(Y_i) < \varepsilon$ for all i, no Y_i can meet both U_P and U_Q. Moreover, since Y covers Z and $P \cup Q \subseteq Z$, there exists indices p and q such that $Y_p \cap U_P \neq \emptyset$ and $Y_q \cap U_Q \neq \emptyset$. Therefore, $A \cap B \cap Y$ has been expressed as the union of two nonempty disjoint closed sets, a contradiction since X is $\operatorname{Un}(Y)$. □

Corollary 3.3. *The continuum X is a dendrite if and only if for each point p in X and for each open set U containing p there exists an open connected set V such that $p \in V \subseteq U$ and X is* $\operatorname{Un}(\bar{V})$.

Proof. If the continuum X is a dendrite, then the existence of the set V follows from the local connectivity and hereditary unicoherence of X.

Conversely, let $\varepsilon > 0$. Then X can be covered by finitely many connected open sets $V_1, V_2, \ldots, V_n$ such that for $i = 1, 2, \ldots, n$, $\operatorname{diam}(\bar{V}_i) < \varepsilon$ and X is $\operatorname{Un}(\bar{V}_i)$. By Theorem 3.2, X is a dendrite. □

The next result is due to Bennett [1].

Theorem 3.4. *Let the continuum X be strongly unicoherent. Then X is aposyndetic at the point p if and only if X is connected im kleinen at the point p.*

It should be noted that Bennett's proof of the above theorem remains valid if the requirement of strong unicoherence is replaced by weak hereditary unicoherence; this observation is made in the following theorem.

Theorem 3.5. *Let the continuum X be weakly hereditarily unicoherent. Then X is aposyndetic at the point p if and only if X is connected im kleinen at the point p.*

Corollary 3.6. *Let the continuum X be weakly hereditarily unicoherent. Then X is aposyndetic if and only if X is locally connected.*

The following theorem is a generalization of Bennett's characterization of dendrites.

Theorem 3.7. *The continuum X is a dendrite if and only if X is aposyndetic and weakly hereditarily unicoherent.*

Proof. If the continuum X is a dendrite, then the aposyndesis and weak hereditary unicoherence of X follow from Bennett's characterization of dendrites and Corollary 2.6.

For the converse, by Bennett's characterization, it suffices to prove that X is strongly unicoherent. Suppose X is not strongly unicoherent. By Theorem 2.1, there exists Y a subcontinuum of X such that X is not Un(Y); therefore, there exist proper subcontinua A and B of X such that $X = A \cup B$ and $A \cap B \cap Y$ is not connected. It follows that there exists a point x such that

$$x \in Y \setminus (A \cap B) \subseteq X \setminus (A \cap B);$$

moreover, $X \setminus (A \cap B)$ is an open subset of X. By Corollary 3.6, X is locally connected; since X is also regular, there exists an open connected subset V of X such that $x \in V \subseteq \bar{V} \subseteq X \setminus (A \cap B)$. If $Z = Y \cup \bar{V}$, then Z in a subcontinuum of X having nonempty interior; therefore, by Theorem 2.5, X is Un(Z). However, the set

$$A \cap B \cap Z = A \cap B \cap (Y \cup \bar{V}) = A \cap B \cap Y$$

is not connected, a contradiction. □

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